Modular Lie Powers and the Solomon descent algebra

Karin Erdmann and Manfred Schocker*

erdmann@maths.ox.ac.uk schocker@maths.ox.ac.uk

Mathematical Institute

24–29 St. Giles'

Oxford OX1 3LB

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Abstract

Let V be an r-dimensional vector space over an infinite field F of prime characteristic p, and let $L_n(V)$ denote the n-th homogeneous component of the free Lie algebra on V. We study the structure of $L_n(V)$ as a module for the general linear group $GL_r(F)$ when $n = pk$ and k is not divisible by p and where $n \geq r$. Our main result is an explicit 1-1 correspondence, multiplicity-preserving, between the indecomposable direct summands of $L_k(V)$ and the indecomposable direct summands of $L_n(V)$ which are not isomorphic to direct summands of $V^{\otimes n}$. The direct summands of $L_k(V)$ have been parametrised earlier, by Donkin and Erdmann. Bryant and Stöhr have considered the case $n = p$ but from a different perspective.

Our approach uses idempotents of the Solomon descent algebras, and in addition a correspondence theorem for permutation modules of symmetric groups.

1 Introduction

Let F be an infinite field of prime characteristic p and let V be an r-dimensional vector space over F. Let $L(V)$ be the free Lie algebra on V and denote its homogeneous component of degree n by $L_n(V)$, for each positive integer n. The group of graded automorphisms of $L(V)$ can be identified with the general linear group $\mathrm{GL}_r(F)$ in such a way that $L_1(V)$ becomes the natural $\mathrm{GL}_r(F)$-module. In this way, $L_n(V)$ becomes a submodule of the n-fold tensor product $V^{\otimes n}$, called the nth Lie power of V. One would like to know the structure of $L_n(V)$ as a module for $\mathrm{GL}_r(F)$. As is well-known, if p does not divide n then $L_n(V)$ is a direct summand of $V^{\otimes n}$. The direct sum decomposition of $L_n(V)$ when p does not divide n was dealt with in [13], generalising naturally the classical theorems for characteristic zero. Degrees divisible by p, however, have largely been a mystery.

Here we study the module structure of $L_n(V)$ when n is divisible by p but not divisible by p^2. Our main result is the following:

Theorem 1. *Let $n = pk$ such that k is not divisible by p and assume $r \geq n$. Then there is a 1-1 correspondence, multiplicity-preserving, between the indecomposable direct summands of $L_k(V)$ and the indecomposable direct summands of $L_n(V)$ which are not isomorphic to direct summands of $V^{\otimes n}$.*

The case $k = 1$ was considered in [6].

The Schur functor relates representations of $\mathrm{GL}_r(F)$ with representations of the symmetric group $\mathcal{S}_n$ for $r \geq n$ (see [17, Chapter 6]). The image under the Schur functor of $L_n(V)$ is the Lie module L_n of $\mathcal{S}_n$ over F which can be described as $L_n = \omega_n F\mathcal{S}_n$ where

$$\omega_n = (1 - \zeta_n)(1 - \zeta_{n-1}) \cdots (1 - \zeta_2) \in F\mathcal{S}_n$$

is the Dynkin operator (see, for instance, [4]). Here ζ_k denotes the descending

k-cycle $(k \, \dots \, 1) \in \mathcal{S}_n$.

Our main tools come from the Solomon descent algebra, a subalgebra of the group algebra of the symmetric group which contains ω_n (see Section 2). An analogous approach was used in [21] to study Lie powers over fields of characteristic zero.

Let $S^p(L_k)$ be the image under the Schur functor of the pth symmetric power $S^p(L_k(V))$ of $L_k(V)$. In Section 4, we shall show that there is a short exact sequence of $\mathcal{S}_n$-modules

$$0 \to L_n \to e_n F\mathcal{S}_n \to S^p(L_k) \to 0$$

where e_n is an idempotent in the Solomon descent algebra.

The middle term is projective, hence the Heller operator Ω gives us a 1-1 correspondence between the non-projective indecomposable direct summands of $S^p(L_k)$ and the non-projective indecomposable direct summands of L_n. The module $S^p(L_k)$ turns out to be a direct summand of a permutation module and can be analysed with modular representation theory tools (see Section 5). As a result, there is a 1-1 correspondence, multiplicity-preserving, between the non-projective indecomposable direct summands of $S^p(L_k)$ and the indecomposable direct summands of L_k. (In fact, this result holds more generally, see Theorem 11.)

This implies:

Theorem 2. *Let k be a positive integer not divisible by p, then there is a 1-1 correspondence, multiplicity-preserving, between the non-projective indecomposable direct summands of L_{pk} and the indecomposable direct summands of L_k.*

For instance, if p is odd then L_{2p} has a unique non-projective indecomposable direct summand, since L_2 has dimension one. A detailed description of this non-projective summand of L_{2p} is given in Theorem 17.

The short exact sequence of $\mathcal{S}_n$-modules has an analogue on the level of $\mathrm{GL}_r(F)$-modules, namely

$$0 \to L_n(V) \to e_n V^{\otimes n} \to S^p(L_k(V)) \to 0.$$

This is shown in Section 6. The modules occuring in this sequence are n-homogeneous polynomial representations of $\mathrm{GL}_r(F)$, that is, they are modules for the Schur algebra $S(r,n)$, see [17, Chapter 2]. For $r \geq n$, the indecomposable direct summands of $V^{\otimes n}$ are precisely the indecomposable modules which are projective and injective as modules for the Schur algebra [12, p. 94]. The middle term, a direct summand of $V^{\otimes n}$, is thus projective and injective as a module for $S(r,n)$. It follows that Ω gives a 1-1 correspondence between the indecomposable direct summands of $S^p(L_k(V))$ which are not projective and injective and the indecomposable direct summands of $L_n(V)$ which are not projective and injective. A 1-1 correspondence between the indecomposable direct summands of $S^p(L_k(V))$ which are not projective and injective and the indecomposable direct summands of $L_k(V)$ is then readily derived and allows us to deduce Theorem 1 (see Proposition 22 and the subsequent remark).

As a further consequence, every indecomposable direct summand of L_{pk} is liftable, hence the formal character of any indecomposable direct summand of $L_{pk}(V)$ is a sum of Schur functions (see Corollary 16).

Using the short exact sequence of $\mathrm{GL}(V)$-modules mentioned above, we also prove a special case of a conjecture of Bryant [7] on Lie resolvents in the final Section 7.

In what follows we shall simply say “summand” for “direct summand” of a module.

2 The descent algebra

Let n be a positive integer. We summarise properties of the descent algebra of the symmetric group $\mathcal{S}_n$. For general reference, see [4, 16, 20, 22]. Note that products $\pi\sigma$ of permutations $\pi, \sigma \in \mathcal{S}_n$ are to be read from left to right: first π, then σ.

Let μ be a composition of n, that is, a finite sequence $(\mu_1, \ldots, \mu_k)$ of positive integers with sum n. We then write $\mathcal{S}_\mu$ for the usual embedding of the direct product $\mathcal{S}_{\mu_1} \times \cdots \times \mathcal{S}_{\mu_k}$ in $\mathcal{S}_n$.

The length of a permutation π in $\mathcal{S}_n$ is the number of inversions of π. Each right coset of $\mathcal{S}_\mu$ in $\mathcal{S}_n$ contains a unique permutation of minimal length. Define X^μ to be the sum in the integral group ring $\mathbb{Z}\mathcal{S}_n$ of all these minimal coset representatives of $\mathcal{S}_\mu$ in $\mathcal{S}_n$. For example, $X^{(n)}$ is the identity of $\mathcal{S}_n$, while $X^{(1,1\ldots,1)}$ is the sum over all permutations in $\mathcal{S}_n$. Due to Solomon [24, Theorem 1], the $\mathbb{Z}$-linear span $\mathcal{D}_n$ of the elements X^μ (μ any composition of n), is a subring of $\mathbb{Z}\mathcal{S}_n$ of rank 2^{n-1}, called the *descent algebra* of $\mathcal{S}_n$. In fact, the elements X^μ form a $\mathbb{Z}$-basis of $\mathcal{D}_n$ and there exist nonnegative integers $c_{\lambda\mu\nu}$ such that

$$X^\lambda X^\mu = \sum_\nu c_{\lambda\mu\nu} X^\nu, \tag{1}$$

for all compositions λ, μ of n. It is well-known that $\omega_n \in \mathcal{D}_n$ and a result of Dynkin-Specht-Wever states that $\omega_n^2 = n\omega_n$ (see, for instance, [4, 15]).

The *Young character* φ^μ of $\mathcal{S}_n$ is induced from the trivial character of $\mathcal{S}_\mu$; that is, $\varphi^\mu(\pi)$ is the number of right cosets of $\mathcal{S}_\mu$ in $\mathcal{S}_n$ which are fixed by right multiplication with π, for any $\pi \in \mathcal{S}_n$. The $\mathbb{Z}$-linear span $\mathcal{C}_n$ of the Young characters φ^μ (μ any composition of n) is a subring of the ring of $\mathbb{Z}$-valued class functions of $\mathcal{S}_n$. In fact, the elements φ^μ form a $\mathbb{Z}$-basis of $\mathcal{C}_n$

and products have the form

$$\varphi^{\lambda}\,\varphi^{\mu}=\sum_{\nu}c_{\lambda\mu\nu}\,\varphi^{\nu} \tag{2}$$

for all compositions λ, μ of n, with the same coefficients as in (1). As a consequence, the $\mathbb{Z}$-linear map $c_n:\mathcal{D}_n\to\mathcal{C}_n$, defined by

$$X^{\mu}\longmapsto\varphi^{\mu} \tag{3}$$

for all compositions μ of n, is an epimorphism of rings. This is the second part of [24, Theorem 1].

Theorem 3. *Let F be a field, then the F-linear span $\mathcal{D}_{n,F}$ of the elements X^{μ} is a subalgebra of the group algebra $F\mathcal{S}_n$, while the F-linear span $\mathcal{C}_{n,F}$ of the F-valued Young characters*

$$\varphi^{\mu,F}:\mathcal{S}_n\to F,\,\pi\mapsto\varphi^{\mu}(\pi)\cdot 1_F$$

is a subalgebra of the algebra of F-valued class functions of $\mathcal{S}_n$. The F-linear map

$$c_{n,F}:\mathcal{D}_{n,F}\to\mathcal{C}_{n,F}$$

sending X^{μ} to $\varphi^{\mu,F}$ for all compositions μ of n, is an epimorphism of algebras.

Indeed, by definition, there are the product formulae (1) and (2) in $\mathcal{D}_{n,F}$ and $\mathcal{C}_{n,F}$, respectively, where the coefficients $c_{\lambda\mu\nu}$ should be read as $c_{\lambda\mu\nu}\cdot 1_F$.

Theorem 4. rad $\mathcal{D}_{n,F}=\ker\,c_{n,F}$.

This is [24, Theorem 3] if F has characteristic zero and [2, Theorem 2] if F has prime characteristic. As a consequence, $\mathcal{D}_{n,F}/\operatorname{rad}\mathcal{D}_{n,F}\cong\mathcal{C}_{n,F}$.

We analyse the algebra $\mathcal{C}_{n,F}$. The conjugacy classes C_{λ} of $\mathcal{S}_n$ are indexed by partitions λ of n, in a natural way: C_{λ} consists of all permutations in $\mathcal{S}_n$ of cycle type λ.

Let λ, μ be partitions of n and p be a prime, then λ and μ are *p-equivalent* if the p-regular parts of π, σ are conjugate in $\mathcal{S}_n$, for each $\pi \in C_\lambda$, $\sigma \in C_\mu$. Note that the cycle type ν of the p-regular part of $\pi \in C_\lambda$ is obtained from λ by replacing each entry $\lambda_i = kp^m$ of λ by the sequence $(k, \ldots, k)$ of length p^m, where $k \geq 1$ and $m \geq 0$ are so chosen that k is not divisible by p. For example, if $p = 2$ and $\lambda = (6, 3, 2)$, then $\nu = (3, 3, 3, 1, 1)$.

The partition μ is *p-regular* if no part of μ occurs more than $p - 1$ times in μ. It is convenient to extend these definitions to the case where $p = 0$, by saying that any partition is 0-regular, and 0-equivalent to itself only.

Then the p-regular partitions form a transversal for the p-equivalence classes of partitions, so that each partition λ is p-equivalent to a unique p-regular partition μ.

For the remainder of this section, F is a field of characteristic p (which might be zero or not). Define $C_{\mu,F}$ to be the union of all conjugacy classes C_λ in $\mathcal{S}_n$ such that λ is p-equivalent to μ, for each p-regular partition μ of n. Let $\mathrm{char}_{\mu,F}$ denote the characteristic function $\mathcal{S}_n \to F$ of $C_{\mu,F}$ (mapping $\pi \in \mathcal{S}_n$ to 1_F or zero according as $\pi \in C_{\mu,F}$ or not).

Proposition 5. *$\mathcal{C}_{n,F}$ is split semisimple. The elements $\mathrm{char}_{\mu,F}$, indexed by p-regular partitions μ of n, form a full set of primitive idempotents in $\mathcal{C}_{n,F}$.*

(see, for instance, [2, Lemma 2 and its proof]) Combining Proposition 5 with Theorem 4, we obtain the following result.

Corollary 6. *$\mathcal{D}_{n,F}$ is a basic algebra with irreducible modules indexed by p-regular partitions of n. In fact, there exists a complete set of mutually orthogonal primitive idempotents*

$$\{\, e_{\mu,F} \mid \mu \text{ } p\text{-regular} \}$$

in $\mathcal{D}_{n,F}$ such that $c_{n,F}(e_{\mu,F}) = \mathrm{char}_{\mu,F}$ for all μ.

(for the second part, see [14, Theorem 44.3].)

The idempotent $e_n = e_{(n),F}$ will be of crucial importance for our study of the modular Lie representations. This section concludes with two observations on e_n. Choose coefficients $a_\mu \in F$ such that $e_n = \sum_\mu a_\mu X^\mu$, where the sum is over compositions μ of n. The image of e_n is $\mathrm{char}_{(n),F}$, thus maps long cycles $\pi \in C_{(n)}$ to 1_F. But $\varphi^{\mu,F}(\pi) = 0$ for all such π whenever $\mu \neq (n)$, while $\varphi^{(n),F}(\pi) = 1_F$. This implies $a_{(n)} = 1_F$, that is

$$e_n = X^{(n)} + \sum_{\mu \neq (n)} a_\mu X^\mu. \tag{4}$$

The second important property of the idempotent e_n is that

$$\dim e_n F\mathcal{S}_n = |C_{(n),F}|. \tag{5}$$

This is a special case of the following result.

Proposition 7. $\dim e_{\mu,F} F\mathcal{S}_n = |C_{\mu,F}|$, *for each p-regular partition μ of n.*

Proof. Let $M_{\mu,F}$ denote the (one-dimensional) irreducible $\mathcal{D}_{n,F}$-module corresponding to $e_{\mu,F}$, for each p-regular partition μ of n. The action of $\alpha \in \mathcal{D}_{n,F}$ on $M_{\mu,F}$ is then scalar multiplication with $c_{n,F}(\alpha)(\pi)$, where $\pi \in C_\mu$. In particular, the family $\{M_{\mu,F}\}$ is defined over $\mathbb{Z}$, since the Young characters take values in $\mathbb{Z}$.

The decomposition matrix of $\mathcal{D}_{n,\mathbb{Q}}$ modulo p is very simple; if λ and μ are partitions of n and the characteristic p of F is positive, then $M_{\lambda,F} \cong M_{\lambda,\mathbb{Z}} \otimes F$ and $M_{\mu,F} \cong M_{\mu,\mathbb{Z}} \otimes F$ are isomorphic as $\mathcal{D}_{n,F}$-modules if and only if λ and μ are p-equivalent. This is due to Atkinson-Pfeiffer-Willigenburg [1, Theorem 4 and Section 4.1] and follows from Theorem 4 and Proposition 5.

If V is an arbitrary $\mathcal{D}_{n,F}$-left module, then the multiplicity $[V : M_{\mu,F}]$ of $M_{\mu,F}$ in a composition series of V is equal to the dimension of $e_{\mu,F}V$, since

$M_{\mu,F}$ has dimension one. It follows that

$$\dim\, e_{\mu,F}F\mathcal{S}_n = [F\mathcal{S}_n : M_{\mu,F}] = \sum_{\lambda}[\mathbb{Q}\mathcal{S}_n : M_{\lambda,\mathbb{Q}}] = \sum_{\lambda}\dim\, e_{\lambda,\mathbb{Q}}\mathbb{Q}\mathcal{S}_n\,,$$

where both sums are taken over all partitions λ of n which are p-equivalent to μ. However, $\dim\, e_{\lambda,\mathbb{Q}}\mathbb{Q}\mathcal{S}_n = |C_\lambda|$ for all partitions λ of n (see, for instance, [22, Lemma 3.2]). This completes the proof. $\square$

3 The Lie module in prime degree

Let F be a field of prime characteristic p. To illustrate our approach, we start with studying the Lie module L_p of the symmetric group $\mathcal{S}_p$ (although this will be generalised later). This module has already been analysed in [6, 10].

For each positive integer n, we consider the sum s_n in $F\mathcal{S}_n$ of all permutations in $\mathcal{S}_n$ as a linear generator of the trivial $\mathcal{S}_n$-module F.

Theorem 8. *There is a short exact sequence of $\mathcal{S}_p$-right modules*

$$0 \longrightarrow L_p \xrightarrow{\alpha} e_pF\mathcal{S}_p \xrightarrow{\beta} F \longrightarrow 0,$$

where α is left multiplication with e_p and β is left multiplication with s_p.

Proof. First let n be an arbitrary positive integer, then a multiplication rule in $\mathcal{D}_n$ (over $\mathbb{Z}$) is

$$X^\mu\omega_n = 0 \text{ whenever } \mu \neq (n) \tag{6}$$

(see [15, §2]), which implies

$$e_n\omega_n = \omega_n + \sum_{\mu\neq(n)} a_\mu X^\mu\omega_n = \omega_n\,, \tag{7}$$

by (4). Thus, in particular, α is an inclusion. Furthermore, the image $\omega_pF\mathcal{S}_p$ of α is contained in the kernel of β, since $s_p\omega_p = X^{(1,\ldots,1)}\omega_p = 0$, by (6).

If $\mu = (\mu_1, \ldots, \mu_l)$ is a composition of p, then the number of summands in X^μ is equal to the number of right cosets of $\mathcal{S}_\mu$ in $\mathcal{S}_p$ which is $\binom{p}{\mu_1 \cdots \mu_l}$. Hence $s_p X^\mu = 0$ in $F\mathcal{S}_p$ whenever $\mu \neq (p)$. This implies $s_p e_p = s_p$, by (4) again. Thus β is onto.

Finally, $\dim L_p = (p-1)!$ is well-known, while $\dim e_p F\mathcal{S}_p = (p-1)! + 1$ follows from (5). Comparing dimensions, completes the proof. □

Corollary 9. *L_p has a unique non-projective indecomposable summand, which is isomorphic to the Specht module associated to the partition $(p-1, 1)$.*

Proof. The module $e_p F\mathcal{S}_p$ is projective, since e_p is an idempotent. The trivial $\mathcal{S}_p$-module is non-projective indecomposable, hence by Theorem 8, the only non-projective indecomposable summand of L_p is isomorphic to the Heller translate $\Omega(F)$ of F (see [3, §1.5]). This is known to be the Specht module mentioned. □

The character of L_p over $\mathbb{Q}$ is known, and also knowing that L_p is the direct sum of a certain Specht module and a projective module, determines uniquely the projective summand. Details have been worked out in [6, Theorem 6.2]; see also [10].

4 Lie modules in degree not divisible by p^2

With a little more input from the descent algebra we will now extend Theorem 8 to the Lie module L_n of $\mathcal{S}_n$ for arbitrary n divisible by p, but not divisible by p^2.

Throughout, F is a field of prime characteristic p and $n = kp$ with k not divisible by p. The *pth symmetrisation* $S^p(L_k)$ of the Lie module for $\mathcal{S}_k$ is a

module for $\mathcal{S}_n$ and is defined as a module induced from the wreath product $\mathcal{S}_k \wr \mathcal{S}_p$, as follows.

For $\alpha \in \mathcal{S}_a$, $\beta \in \mathcal{S}_b$, define $\alpha\#\beta \in \mathcal{S}_{(a,b)} \subseteq \mathcal{S}_{a+b}$ in the natural way: $\alpha\#\beta$ maps i to $i\alpha$ if $i \leq a$, and to $(i-a)\beta + a$ otherwise. If, additionally, $\gamma \in \mathcal{S}_c$, then $(\alpha\#\beta)\#\gamma = \alpha\#(\beta\#\gamma)$, as is readily seen. Using linearity, we define

$$\omega^{(k,\dots,k)} := \omega_k\#\cdots\#\omega_k \in F(\mathcal{S}_k\#\cdots\#\mathcal{S}_k) = F\mathcal{S}_{(k,\dots,k)} \quad (p \text{ factors}).$$

Now assume $\pi \in \mathcal{S}_p$. Let $\pi^{[k]}$ be the element of $\mathcal{S}_{kp}$ which permutes the p successive blocks of size k in $\{1,\dots,kp\}$ according to π. More explicitly, set $(ik-j)\pi^{[k]} = (i\pi)k - j$ for all $i \in \{1,\dots,p\}$ and $j \in \{0,\dots,k-1\}$.

The map $\pi \mapsto \pi^{[k]}$ extends linearly to an embedding of $F\mathcal{S}_p$ in $F\mathcal{S}_{kp}$, since $(\pi\sigma)^{[k]} = \pi^{[k]}\sigma^{[k]}$ for all $\pi, \sigma \in \mathcal{S}_k$. Furthermore, $\mathcal{S}_p^{[k]}\mathcal{S}_{(k,\dots,k)}$ is isomorphic to the wreath product $\mathcal{S}_k \wr \mathcal{S}_p$, since

$$(\alpha_1\#\cdots\#\alpha_p)(\beta_1\#\cdots\#\beta_p) = (\alpha_1\beta_1)\#\cdots\#(\alpha_p\beta_p) \tag{8}$$

and

$$\pi^{[k]}(\alpha_1\#\cdots\#\alpha_p) = (\alpha_{1\pi}\#\cdots\#\alpha_{p\pi})\pi^{[k]} \tag{9}$$

for all $\pi \in \mathcal{S}_p$ and $\alpha_1,\dots,\alpha_p,\beta_1,\dots,\beta_p \in \mathcal{S}_k$.

We define now the pth symmetrisation of L_k by

$$S^p(L_k) = s_p^{[k]}\omega^{(k,\dots,k)}F\mathcal{S}_n\,.$$

We shall see after Corollary 20 that this is isomorphic to the image of the pth symmetric power $S^p(L_k(V))$ of $L_k(V)$ under the Schur functor.

The main result of this section is:

Theorem 10. *There is a short exact sequence of $\mathcal{S}_n$-right modules*

$$0 \longrightarrow L_n \xrightarrow{\alpha} e_nF\mathcal{S}_n \xrightarrow{\beta} S^p(L_k) \longrightarrow 0,$$

where α is left multiplication with e_n and β is left multiplication with $X^{(k,\dots,k)}$.

This reduces to Theorem 8 in case $k = 1$. Note that dim $S^p(L_k) = |C_{(k,\dots,k)}|$, so that

$$\dim e_n F\mathcal{S}_n = |C_{(n)}| + |C_{(k,\dots,k)}| = \dim L_n + \dim \mathcal{S}_p(L_k),$$

by (5).

From now on, we write κ for the composition $(k, \dots, k)$ of $n = kp$. Before we prove Theorem 10, let us recall some more multiplication rules for the descent algebra.

If $\lambda = (\lambda_1, \dots, \lambda_l)$ and ν are compositions of n, then write $\nu \le \lambda$ if there is a composition $\nu^{(i)}$ of λ_i for all $i \le l$ such that ν is equal to the concatenation $(\nu^{(1)}, \dots, \nu^{(l)})$. For example, $(1,2,3,2,1,2) \le (3,3,5)$.

Concerning the coefficients $c_{\lambda\mu\nu}$ in (1), there is the restriction

$$c_{\lambda\mu\nu} = 0 \text{ unless } \nu \le \lambda \tag{10}$$

[2, Lemma 1(i)]. Furthermore, for any composition $\mu = (\mu_1, \dots, \mu_l)$ of n with each part divisible by k,

$$c_{\kappa\mu\kappa} = \binom{p}{\mu_1/k \cdots \mu_l/k}. \tag{11}$$

This follows directly from the combinatorial description of $c_{\lambda\mu\nu}$ given in [2, Eq. (1.1)], for example. Finally,

$$X^\mu \omega^\kappa = 0 \text{ unless } \kappa \le \mu \tag{12}$$

and

$$X^\kappa \omega^\kappa = s_p^{[k]} \omega^\kappa \tag{13}$$

(see [16, Theorem 2.1]). It should be mentioned that the equations (12) and (13) are much deeper than the equations (10) and (11).

Proof of Theorem 10. We have already seen that $e_n\omega_n = \omega_n$ (see (7)), so α is an inclusion. Furthermore, $X^\kappa\omega_n = 0$ as in the proof of Theorem 8, by (6), so L_n is contained in the kernel of β. It remains to show that $X^\kappa e_n F\mathcal{S}_n$ contains $S^p(L_k)$, for then a comparison of dimensions completes the proof.

But, for each composition μ of n, (12) yields $X^\mu\omega^\kappa = 0$ unless $\kappa \leq \mu$ (that is, unless each part of μ is divisible by k). In this case,

$$\begin{aligned} X^\kappa X^\mu \omega^\kappa &= \sum_{\nu\leq\kappa} c_{\kappa\mu\nu} X^\nu\omega^\kappa, \quad \text{by (1), (10)} \\ &= c_{\kappa\mu\kappa} X^\kappa\omega^\kappa, \quad \text{by (12)} \\ &= \binom{p}{\mu_1/k \cdots \mu_l/k} X^\kappa\omega^\kappa, \quad \text{by (11).} \end{aligned}$$

As a consequence, $X^\kappa e_n\omega^\kappa = X^\kappa\omega^\kappa = s_p^{[k]}\omega^\kappa$ in $F\mathcal{S}_n$, by (4) and (13), which implies $X^\kappa e_n F\mathcal{S}_n \supseteq X^\kappa e_n\omega^\kappa F\mathcal{S}_n = s_p^{[k]}\omega^\kappa F\mathcal{S}_n = S^p(L_k)$, and we are done. □

As in the special case where $k = 1$, Theorem 10 gives the 1-1 correspondence $U \mapsto \Omega(U)$ between the non-projective indecomposable summands of $S^p(L_k)$ and those of L_n, since $e_n F\mathcal{S}_n$ is a projective $\mathcal{S}_n$-module.

5 Non-projective indecomposable summands of symmetrised modules

Let F be a field of prime characteristic p and k a positive integer not divisible by p. Bearing in mind Theorem 10, we are aiming at a parametrisation of the non-projective indecomposable summands of $S^p(L_k)$.

We shall consider, more generally, an arbitrary idempotent $e \in F\mathcal{S}_k$ (instead of $\frac{1}{k}\,\omega_k$) and the corresponding right ideal $U = eF\mathcal{S}_k$ of $F\mathcal{S}_k$. Let $n = kp$.

Generalising the definition of $S^p(L_k)$ in Section 4, we let the pth symmetrisation of U be the $F\mathcal{S}_n$-module

$$S^p(U) = s_p^{[k]} e^{\#p} F\mathcal{S}_n\,,$$

where $e^{\#p} = e\#\cdots\#e$ (p factors). The aim is to prove

Theorem 11. *There is a 1-1 correspondence, multiplicity-preserving, between the non-projective indecomposable summands of $S^p(U)$ and the indecomposable summands of U.*

We write $H = \mathcal{S}_k \wr \mathcal{S}_p$ (which is taken as the subgroup $\mathcal{S}_p^{[k]}\mathcal{S}_{(k,\dots,k)}$ of $\mathcal{S}_n$).

A crucial step towards a proof of Theorem 11 is the following observation.

Proposition 12. *We have $S^p(U) \cong (F\otimes_{F\mathcal{S}_p} e^{\#p}FH)\otimes_{FH} F\mathcal{S}_n$. In particular, $S^p(U)$ is a summand of a permutation module of $\mathcal{S}_n$.*

Proof. The element $e^{\#p}$ is an idempotent in H, by (8), which commutes with every element of $\mathcal{S}_p \subseteq H$, by (9). So $FH = e^{\#p}FH \oplus (1-e^{\#p})FH$ as a $(F\mathcal{S}_p, FH)$ bimodule. Hence $F\otimes_{F\mathcal{S}_p} e^{\#p}FH$ is a summand of $F\otimes_{F\mathcal{S}_p} FH$, which is a permutation module of H, and therefore $(F\otimes_{F\mathcal{S}_p} e^{\#p}FH)\otimes_{FH} F\mathcal{S}_n$ is a summand of a permutation module of $\mathcal{S}_n$. We want to identify this summand with $S^p(U)$.

In general, for a subgroup Y of a finite group X and an idempotent f of FX which commutes with all elements of Y we have

$$gFY \otimes_{FY} fFX \cong gfFX$$

for all $g \in FY$, since fFX is a projective left FY-module. We apply this twice, the first time with $Y = \mathcal{S}_p$, $X = H$, $f = e^{\#p}$ and $g = s_p^{[k]}$ and the second time with $Y = H$, $X = \mathcal{S}_n$, $f = 1$ and $g = s_p^{[k]}e^{\#p}$. This yields

$$(F\otimes_{F\mathcal{S}_p} e^{\#p}FH)\otimes_{FH} F\mathcal{S}_n \cong s_p^{[k]}e^{\#p}FH\otimes_{FH} F\mathcal{S}_n \cong s_p^{[k]}e^{\#p}F\mathcal{S}_n = S^p(U)$$

as asserted. $\square$

For an arbitrary finite group X, an X-module W is said to be a *p-permutation module* if, for every p-subgroup P of X, there is a P-invariant basis of W. Equivalently, W is a summand of a permutation module of X.

The above proposition implies that $S^p(U)$ is a p-permutation module for $\mathcal{S}_n$. In order to parametrise the non-projective indecomposable summands of $S^p(U)$, it is natural to use their vertices.

Recall that for a finite group X, any indecomposable FX-module Q has a vertex. This is, by definition, a subgroup Y of X such that Q is Y-projective and which is minimal with this property. (A module is Y-projective if it is a summand of $S \otimes_{FY} FX$ for some FY-module S). Any two vertices are conjugate in X, and moreover they are p-subgroups [3, Proposition 3.10.2]. The modules with vertex $\{1\}$ are precisely the projective modules.

By Proposition 12, $S^p(U)$ is $\mathcal{S}_p$-projective. A Sylow p-subgroup D of $\mathcal{S}_p$ is cyclic of order p. Hence an indecomposable summand of $S^p(U)$ is non-projective if and only if it has vertex D.

Lemma 13. *There is a 1-1 correspondence, multiplicity-preserving, between the indecomposable summands of $S^p(U)$ with vertex D and the indecomposable summands of M with vertex D where*

$$M = F \otimes_{F\mathcal{S}_p} e^{\#p} FH.$$

Proof. Recall that $S^p(U) \cong M \otimes_{FH} F\mathcal{S}_n$ where $H = \mathcal{S}_k \wr \mathcal{S}_p$. To prove the statement it is sufficient to establish the following.

If X is an indecomposable summand of M with vertex D then $X \otimes_{FH} F\mathcal{S}_n$ has a unique indecomposable summand $\tilde{X}$ with vertex D; and moreover $X \mapsto \tilde{X}$ is a 1-1 correspondence.

Let $N_1 = N_H(D)$ and $N = N_{\mathcal{S}_n}(D)$. The Green correspondence [3, Theorem 3.12.2] provides us with a 1-1 correspondence between the indecompos-

able FH-modules (or $F\mathcal{S}_n$-modules) with vertex D and the indecomposable FN_1-modules (or FN-modules) with vertex D. So we are done if we can show that if Q is the Green correspondent of X in N_1 then $Q \otimes_{FN_1} FN$ is indecomposable, and that non-isomorphic Green correspondents induce to non-isomorphic FN-modules. (The module $Q \otimes_{FN_1} FN$ has then automatically vertex D.)

Since X is a p-permutation module with vertex D, one knows that D acts trivially on Q and moreover Q is indecomposable projective as a module for N_1/D. Then $Q \otimes_{FN_1} FN$ is still trivial as a module for D, and it is projective as a module for N/D. To complete the proof we exploit the general fact that indecomposable projective modules are in 1-1 correspondence with their simple quotients.

Analysing the groups N_1 and N we will show below (in the following Lemma) that

$$N = N_1 B$$

where B is a p-group which is normal in N.

Recall that a normal p-subgroup acts trivially on all simple modules, therefore we have a 1-1 correspondence between simple modules of N and simple modules of N_1, by restriction. By Frobenius reciprocity, it follows that

$$\mathrm{Hom}_N(Q \otimes_{FN_1} FN, L) \cong \mathrm{Hom}_{N_1}(Q, L)$$

for any simple N-module L and this is F precisely if L is the simple quotient of Q, and zero otherwise. $\square$

Lemma 14. *Let $N_1 = N_H(D)$ and $N = N_{\mathcal{S}_n}(D)$. Then $N = N_1 B$ where B is a p-group which is normal in N. Moreover N_1/D is isomorphic to the direct product of $\mathcal{S}_k$ with a cyclic group of order $p-1$.*

Proof. Observe first that the centraliser $C_H(D)$ of D in H is isomorphic to a direct product $\mathcal{S}_k \times D$. We view this in two ways,

(i) as a subgroup of $\mathcal{S}_k \wr \mathcal{S}_p$ where the first factor of the above direct product is the diagonal $\Delta(\mathcal{S}_k)$ in the base group; and

(ii) as a subgroup of $N_{\mathcal{S}_p}(C_p) \wr \mathcal{S}_k$ where the second factor of the direct product above is contained in the diagonal of the base group.

We get from (i) that $N_1 \cong \Delta(\mathcal{S}_k) \times N_{\mathcal{S}_p}(D)$. Moreover, we get from (ii) that $C_{\mathcal{S}_n}(D)$ is the semi-direct product of B with $\mathcal{S}_k$ where B is the base group of $C_p \wr \mathcal{S}_k$, and that N is generated by $C_{\mathcal{S}_n}(D)$ and $N_{\mathcal{S}_p}(D)$ (contained in the base group) which acts diagonally. Hence B is normal in N and $N = BN_1$.

The description of N_1 implies directly the statement on N_1/D. $\square$

We will apply Broué's correspondence theorem for p-permutation modules which is described now; here X can be an arbitrary finite group again. Assume W is a p-permutation module of X and P is a p-subgroup of X. Set

$$W(P) := W^P / \sum_{Q<P} \mathrm{Tr}_Q^P(W^Q),$$

where W^R denotes the space of fixed points in W of any subgroup R of X and where Tr_Q^P is defined as $\mathrm{Tr}_Q^P(m) = \sum_i mg_i$, the sum taken over a transversal of Q in P. Then $W(P)$ is a module for $N_G(P)$ on which P acts trivially, and hence is a module for the factor group $N_G(P)/P$. As a vector space, $W(P)$ is isomorphic to the span of the fixed points of P in a given permutation basis.

In [5], Broué proved that there is a 1-1 correspondence, multiplicity-preserving, between the indecomposable summands of W with vertex P and the indecomposable summands of $W(P)$ which are projective as $N_G(P)/P$-modules.

Applied to the p-permutation module M of H and combined with Lemma 13, this result implies that there is a 1-1 correspondence between non-projective indecomposable summands of $S^p(U)$ and the indecomposable summands of $M(D)$ which are projective as $N_H(D)/D$-modules. This group is isomorphic to the direct product of $\mathcal{S}_k$ with a cyclic group of order $p-1$ whose group

algebra over F is semi-simple with 1-dimensional simple modules. The proof of Theorem 11 can thus be completed using the following result.

Lemma 15. *The N_1/D-module $M(D)$ is isomorphic to U as an $\mathcal{S}_k$-module. Moreover, the cyclic group of order $p-1$ acts trivially on $M(D)$.*

Proof. Let B and B' be bases of $U = eF\mathcal{S}_k$ and $(1-e)F\mathcal{S}_k$, respectively, so that $B \cup B'$ is a basis of $F\mathcal{S}_k$. Then the induced module $F \otimes_{F\mathcal{S}_p} FH$ has basis $\{\, s_p \otimes (v_1 \# \dots \# v_p) \,|\, v_1, \dots, v_p \in B \cup B' \,\}$. It follows that

$$\{\, s_p \otimes (b_1 \# \dots \# b_p) \,|\, b_1, \dots, b_p \in B \,\}$$

is a basis of M where $M = F \otimes_{F\mathcal{S}_p} e^{\# p} FH$. This is a permutation basis under the action of $\mathcal{S}_p$.

If, in particular, $\zeta \in D$ is a p-cycle which in its action on kp points permutes the supports of the factors cyclically, then

$$\Big(s_p \otimes (b_1 \# b_2 \# \dots \# b_p)\Big)\zeta = s_p \otimes (b_p \# b_1 \# \dots \# b_{p-1})$$

for all $b_1, \dots, b_p \in B$. We deduce that a basis vector of M is fixed by D if and only if it is of the form $s_p \otimes (b \# b \# \dots \# b)$ for some $b \in B$. Such element is also fixed under the cyclic group of order $p-1$ normalising D in $\mathcal{S}_p$ (which proves the last part of the Lemma).

As a consequence, there is an obvious vector space isomorphism $\psi : U \to M(D)$ taking b to the coset of $s_p \otimes (b \# b \# \dots \# b)$ for all $b \in B$. We want to compare the actions of $\mathcal{S}_k$. On the one hand, we have for $\pi \in \mathcal{S}_k$

$$\psi(b\pi) = \psi\Big(\sum_{b' \in B} c(b,b')\, b'\Big) = \sum_{b' \in B} c(b,b')\, \psi(b')$$

with coefficients $c(b,b') \in F$. On the other hand,

$$\psi(b)\pi = s_p \otimes (b\pi \# b\pi \# \dots \# b\pi) = \sum_{b' \in B} c(b,b')^p\; s_p \otimes (b' \# \dots \# b') + (*)$$

where $(*)$ belongs to the span of orbit sums with orbits of size > 1 and which is zero in $M(D)$.

So $M(D)$ is isomorphic to the Frobenius twist of U, obtained by composing the corresponding matrix representation with the map

$$\Big(c(b,b')\Big) \mapsto \Big(c(b,b')^p\Big).$$

But since all projective modules for symmetric groups are defined over the prime field, for any projective module P, its Frobenius twist is isomorphic to P. So the Lemma is proved. $\square$

Combining Theorem 11 with Theorem 10 and applying the Heller operator, yields Theorem 2 mentioned in the introduction.

Recall that an $F\mathcal{S}_n$-module M is *liftable* if there exists an indecomposable $\mathcal{O}\mathcal{S}_n$-module $\tilde{M}$ whose reduction modulo p is M; here $\mathcal{O}$ is a complete discrete valuation ring with residue field F. The above considerations also have the following consequence.

Corollary 16. *Let k be a positive integer not divisible by p, then any non-projective indecomposable summand of L_{pk} is liftable and has an associated complex character.*

Proof. Any p-permutation module is liftable, by Scott's theorem [3, 3.11.3], hence every indecomposable summand of $S^p(L_k)$ is liftable. Furthermore, an Ω-translate of a liftable module is liftable, by a standard argument. The claim follows from Theorem 10 and Proposition 12. $\square$

In concluding this section, we give a more detailed account on the Lie module L_{2p}, for any odd prime p. This is a case where the modular representation theory of the symmetric group is sufficiently well understood, and we have a complete description of $S^p(L_2)$.

Theorem 17. *Let p be an odd prime, then L_{2p} has a unique non-projective indecomposable summand. It belongs to the principal block and has character*

$$\begin{aligned}&\chi^{(2,1^{2p-2})}+\chi^{(3,2^{p-2},1)}\\ &+\sum_{i=2}^{(p-1)/2}\left(\chi^{(2i+1,2i,2^{p-2i-1},1)}+\chi^{((2i-1)^2,2^{p-2i+1})}+\chi^{(2i,2i-1,2^{p-2i},1)}\right),\end{aligned}$$

where χ^μ denotes the irreducible character of $\mathcal{S}_{2p}$ corresponding to μ, for any partition μ of $2p$.

Since the character of L_{2p} is known, one knows the character of the projective part of L_{2p} as well. Since a projective module is determined by its character one can deduce the complete direct sum decomposition of L_{2p}, at least in principle.

The proof of Theorem 17 builds on the following result on $S^p(L_2)$.

Proposition 18. *Let p be an odd prime, then $S^p(L_2)$ has a unique non-projective indecomposable summand Q, namely its principal block component. The character of Q is*

$$\chi^{(1^{2p})}+\sum_{i=1}^{(p-1)/2}\chi^{((2i+1)^2,2^{p-2i+1})}.$$

Proof. Let $H=\mathcal{S}_2\wr\mathcal{S}_p$ and $M=F\otimes_{F\mathcal{S}_p}\omega^{(2,\dots,2)}FH$. We denote the sign representation of $F\mathcal{S}_{2p}$ by sgn, then M is isomorphic to the sign representation $\mathrm{sgn}\,|_{FH}$ of FH, since the idempotent $\frac{1}{2}\omega_2$ generates the sign representation of $\mathcal{S}_2$. Applying Proposition 12 and the tensor identity, it follows that

$$S^p(L_2)\otimes\mathrm{sgn}\cong(M\otimes\mathrm{sgn}\,|_{FH})\otimes_{FH}F\mathcal{S}_n=F\otimes_{FH}F\mathcal{S}_n\,.$$

By definition, H is the centraliser of a fixed point free involution, τ say, hence the module $S^p(L_2)\otimes\mathrm{sgn}$ is isomorphic to the permutation module of the symmetric group $\mathcal{S}_{2p}$ on the conjugacy class of τ.

This module was studied in detail by Wildon [25, Chapter 5]. The component in the principal block is indecomposable, and this is preserved by tensoring with the sign representation. However, by Theorem 11, $S^p(L_2)$ has a unique non-projective indecomposable summand Q. Its Green correspondent belongs to the principal block, hence so does Q. This proves the first claim.

The character of $S^p(L_2)$ is $\sum \chi^\mu$, where the sum is taken over all μ such that the multiplicity of each part of μ is even (see, for example, [25, Theorem 4.1.1]). In order to determine its principal block component, it is convenient to use abacus notation for characters (see [19, p. 78], or [23]). Any partition λ of $2p$ can be represented on an abacus with p runners and with $2p$ beads. Then λ lies in the principal block if and only if the abacus display has two gaps (counting the gaps on each runner which are above the last bead). We write $\langle\, i, j\,\rangle$ if the gaps are on runner(s) i, j and write $\langle\, i\,\rangle$ if there is a gap of size 2 on runner i. These give a complete list of partitions whose character belongs to the principal block.

For example, if $p = 5$, then $\langle\, 1, 3\,\rangle$ denotes the abacus display

$$\begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \cdot & \bullet & \cdot & \bullet & \bullet \\ \bullet & \cdot & \bullet & \cdot & \cdot \end{pmatrix}.$$

This represents the partition $\lambda = (3, 2, 2, 2, 1)$ (count the number of gaps before each bead, reading row-wise from top to bottom).

By a straightforward case-by-case analysis one now shows that the character of Q is

$$\langle\, 1, 1\,\rangle + \sum_{i=1}^{(p-1)/2} \langle\, 2i, 2i+1\,\rangle \tag{14}$$

and this translates directly into the statement. $\square$

Proof of Theorem 17. By Theorem 10 and Proposition 18, the Lie module L_{2p} has a unique non-projective indecomposable summand which belongs to the principal block, namely the Heller translate of Q.

We use abacus notation for characters and irreducible modules in the principal block component again. From [25], Q has top composition factors

$$\bigoplus_{i=1}^{p-1/2} D(\langle\, 2i+1, 2i, \,\rangle).$$

The decomposition matrix of the block is known [19]. The projective cover of $D(\langle\, 2i+1, 2i, \,\rangle)$ has character

$$\langle\, 2i+1, 2i\,\rangle + \langle\, 2i+1, 2i-1\,\rangle + \langle\, 2i-1, 2i-2\,\rangle + \langle\, 2i, 2i-2\,\rangle$$

if $i \geq 2$, and $\langle\, 3,2\,\rangle + \langle\, 3,1\,\rangle + \langle\, 2,2\,\rangle + \langle\, 1,1\,\rangle$ if $i = 1$. Comparison with the character of Q as given in (14), shows that the character of the non-projective indecomposable summand of L_{2p} is

$$\langle\, 2,2\,\rangle + \langle\, 3,1\,\rangle + \sum_{i=2}^{(p-1)/2} \Big(\langle\, 2i+1, 2i-1\,\rangle + \langle\, 2i-1, 2i-2\,\rangle + \langle\, 2i, 2i-2\,\rangle\Big),$$

which readily translates into the claim. □

For general $n = kp$ with $k > 2$ not divisible by p, one does not know the representation theory of symmetric group and the character of $S^p(L_k)$ in sufficient detail in order to derive a result like Theorem 17 in this way.

6 Lie powers in degrees not divisible by p^2

Let F be an infinite field of prime characteristic p and let V be an r-dimensional vector space over F. In this section, we consider Lie powers $L_{pk}(V)$ of V where k is not divisible by p.

If n is an arbitrary positive integer, then $V^{\otimes n}$ is an $(\mathcal{S}_n, \mathrm{GL}(V))$ bimodule. The action of $\mathrm{GL}(V)$ from the right is the diagonal action, while for the $\mathcal{S}_n$-action from the left, we have

$$\pi(v_1 \otimes \cdots \otimes v_n) = v_{1\pi} \otimes \cdots \otimes v_{n\pi}$$

for all $\pi \in \mathcal{S}_n$ and $v_1, \ldots, v_n \in V$.

For $r \geq n$, the Schur algebra $S(r,n)$ contains an idempotent ξ such that the algebra $\xi S(r,n)\xi$ is canonically isomorphic to the group algebra $F\mathcal{S}_n$ and we identify these algebras [17, (6.1d)][1]. The Schur functor (denoted by f in [17]) takes an $S(r,n)$-module M to the $F\mathcal{S}_n$-module $M\xi$. The tensor space $V^{\otimes n}$ is isomorphic to $\xi S(r,n)$ as an $(F\mathcal{S}_n, S(r,n))$ bimodule [17, (6.4f)]. Hence the left adjoint of the Schur functor (which we denote by $g_\otimes$) takes a right $F\mathcal{S}_n$-module U to $U \otimes_{F\mathcal{S}_n} V^{\otimes n}$.

The functor $g_\otimes$ is right exact but not exact. However, it is possible to lift the short exact sequence of $\mathcal{S}_n$-modules given in Theorem 10 to a short exact sequence of $\mathrm{GL}(V)$-modules and to parametrise the indecomposable summands of $L_{pk}(V)$ which are not summands of $V^{\otimes n}$ accordingly.

We will use the following tool, to translate between the symmetric group and the general linear group.

Lemma 19. *Let n be a positive integer and assume that*

$$0 \longrightarrow fF\mathcal{S}_n \xrightarrow{\alpha} eF\mathcal{S}_n \longrightarrow U \longrightarrow 0$$

is a short exact sequence of $\mathcal{S}_n$-right modules, where $e, f \in F\mathcal{S}_n$ satisfy $ef = f$ and $e^2 = e$, and where α is left multiplication with e. Then there is a short exact sequence of $\mathrm{GL}(V)$-modules

$$0 \longrightarrow fV^{\otimes n} \longrightarrow e \otimes_{F\mathcal{S}_n} V^{\otimes n} \longrightarrow U \otimes_{F\mathcal{S}_n} V^{\otimes n} \longrightarrow 0.$$

[1]Note that the roles of r and n here and left and right action are exchanged in comparison with [17].

Proof. There is an embedding of $\mathrm{GL}(V)$-modules $\alpha' : fV^{\otimes n} \to eV^{\otimes n}$ provided by left action of e. Furthermore, application of $g_\otimes$ to the short exact sequence of $\mathcal{S}_n$-modules gives the exact sequence

$$f \otimes_{F\mathcal{S}_n} V^{\otimes n} \xrightarrow{\gamma} e \otimes_{F\mathcal{S}_n} V^{\otimes n} \longrightarrow U \otimes_{F\mathcal{S}_n} V^{\otimes n} \longrightarrow 0$$

of $\mathrm{GL}(V)$-modules. The multiplication map $F\mathcal{S}_n \otimes_{F\mathcal{S}_n} V^{\otimes n} \to F\mathcal{S}_n V^{\otimes n}$ restricts to epimorphisms $\beta : f \otimes_{F\mathcal{S}_n} V^{\otimes n} \to fV^{\otimes n}$ and $\delta : e \otimes_{F\mathcal{S}_n} V^{\otimes n} \to eV^{\otimes n}$ and gives rise to a commutative diagram

$$\begin{tikzcd}
 & f \otimes_{F\mathcal{S}_n} V^{\otimes n} \arrow[r, "\gamma"] \arrow[d, "\beta"'] & e \otimes_{F\mathcal{S}_n} V^{\otimes n} \arrow[r] \arrow[d, "\delta"'] & U \otimes_{F\mathcal{S}_n} V^{\otimes n} \arrow[r] & 0 \\
0 \arrow[r] & fV^{\otimes n} \arrow[r, "\alpha'"'] & eV^{\otimes n} & &
\end{tikzcd}$$

But δ is an isomorphism, since e is an idempotent. It follows that $\ker \gamma = \ker \beta$ and that $\delta^{-1}\alpha'$ is an embedding of $fV^{\otimes n}$ into $e \otimes_{F\mathcal{S}_n} V^{\otimes n}$ such that $\delta^{-1}\alpha'\beta = \gamma$. □

Corollary 20. *If $n = pk$ such that k is not divisible by p, then there is a short exact sequence of* $\mathrm{GL}(V)$*-modules*

$$0 \longrightarrow L_n(V) \longrightarrow e_n \otimes_{F\mathcal{S}_n} V^{\otimes n} \longrightarrow S^p(L_k(V)) \longrightarrow 0,$$

where $S^p(L_k(V))$ denotes the p-th symmetric power of $L_k(V)$.

Proof. This follows from Theorem 10 and Lemma 19, applied to $f = \omega_n$, $e = e_n$ and $U = S^p(L_k)$, we just have to identify the cokernel. Let $H = \mathcal{S}_k \wr \mathcal{S}_p$, then $\omega^{(k,\ldots,k)}$ is (up to the factor $1/k^p$) an idempotent in FH. Hence, by Proposition 12,

$$\begin{aligned}
S^p(L_k) \otimes_{F\mathcal{S}_n} V^{\otimes n} &\cong \Big((F \otimes_{F\mathcal{S}_p} \omega^{(k,\ldots,k)} FH) \otimes_{FH} F\mathcal{S}_n \Big) \otimes_{F\mathcal{S}_n} V^{\otimes n} \\
&\cong (F \otimes_{F\mathcal{S}_p} \omega^{(k,\ldots,k)} F\mathcal{S}_n) \otimes_{F\mathcal{S}_n} V^{\otimes n}
\end{aligned}$$

$$
\begin{aligned}
&\cong F \otimes_{F\mathcal{S}_p} \omega^{(k,\ldots,k)} V^{\otimes n} \\
&\cong F \otimes_{F\mathcal{S}_p} L_k(V)^{\otimes p} \\
&\cong S^p(L_k(V))
\end{aligned}
$$

as desired. $\square$

Note that, as the proof shows, the image under $g_\otimes$ of $S^p(L_k)$ is isomorphic to $S^p(L_k(V))$. Therefore the image under the Schur functor f of $S^p(L_k(V))$ is isomorphic to $S^p(L_k)$, since $f \circ g_\otimes$ is naturally equivalent to the identity.

To prove Theorem 1, we will use the following general observations on the functor $g_\otimes$.

Proposition 21. *Let n be a positive integer such that $r \geq n$. Then*

1. *If M is an indecomposable $F\mathcal{S}_n$-module, then $M \otimes_{F\mathcal{S}_n} V^{\otimes n}$ is indecomposable.*
2. *If M_1 and M_2 are indecomposable $F\mathcal{S}_n$-modules which are not isomorphic, then $M_1 \otimes_{F\mathcal{S}_n} V^{\otimes n}$ and $M_2 \otimes_{F\mathcal{S}_n} V^{\otimes n}$ are not isomorphic.*

Proof. Let $S = S(r, n)$ be the Schur algebra and let f be the Schur functor. Recall that $g_\otimes$ is left adjoint to f and that this functor followed by the Schur functor is naturally equivalent to the identity. So

$$\mathrm{Hom}_S(g_\otimes M, g_\otimes M) \cong \mathrm{Hom}_{F\mathcal{S}_n}(M, f \circ g_\otimes M) \cong \mathrm{Hom}_{F\mathcal{S}_n}(M, M),$$

by adjointness, even an isomorphism of algebras. It follows that M is indecomposable if and only if $g_\otimes M$ is indecomposable.

To prove the second part, let M_1 and M_2 so that $g_\otimes M_1$ and $g_\otimes M_2$ are isomorphic, then also $M_1 \cong f \circ g_\otimes M_1 \cong f \circ g_\otimes M_2 \cong M_2$. $\square$

For $r \geq n$, any summand of the tensor space $V^{\otimes n}$ is projective and injective as a module for the Schur algebra $S(r,n)$ (see, for example, [12, p. 94]). As a consequence, the Heller operator Ω gives a 1-1 correspondence between indecomposable non-projective quotients of $V^{\otimes n}$ and indecomposable non-injective submodules of $V^{\otimes n}$.

Furthermore, if M is a quotient of $V^{\otimes n}$ then M is not projective if and only if it is not projective and injective. One direction is clear. For the converse, assume M is projective then it is a summand of $V^{\otimes n}$ and hence is also injective. Similarly, a submodule of $V^{\otimes n}$ is not injective if and only if it is not projective and injective.

Proposition 22. *Let $n = pk$ such that k is not divisible by p and assume $r \geq n$.*

1. *The functor $g_{\otimes}$ gives a 1-1 correspondence, multiplicity-preserving, between the non-projective indecomposable summands of the $F\mathcal{S}_n$-module $S^p(L_k)$ and the indecomposable summands of the $S(r,n)$-module $S^p(L_k(V))$ which are not projective and injective.*

2. *The Heller operator Ω gives a 1-1 correspondence, multiplicity-preserving, between the indecomposable summands of $S^p(L_k(V))$ which are not projective and injective and the indecomposable summands of $L_n(V)$ which are not projective and injective, both as modules for $S(r,n)$.*

Proof. 1. By the preceding proposition we get the 1-1 correspondence between indecomposable summands of $S^p(L_k)$ and $S^p(L_k(V))$. Moreover, an indecomposable $F\mathcal{S}_n$-module M is projective if and only if $g_{\otimes}M$ is an indecomposable summand of $V^{\otimes n}$ since $g_{\otimes}$ takes $F\mathcal{S}_n$ to $V^{\otimes n}$. This completes the proof of the first part.

2. Consider the short exact sequence

$$0 \to L_n(V) \to e_n F\mathcal{S}_n \otimes V^{\otimes n} \to S^p(L_k(V)) \to 0.$$

The middle is isomorphic to a summand of $V^{\otimes n}$. As explained above, Ω induces a 1-1 correspondence as stated in the second part. □

It was shown in [13] that there is a 1-1 correspondence between the indecomposable summands of L_k and those of $L_k(V)$. Explicitly, for k not divisible by p, the module $L_k(V)$ is a summand of $V^{\otimes k}$. The indecomposable summands of $V^{\otimes k}$ are parametrised by p-regular partitions λ of k, as $T(\lambda)$, where $T(\lambda)$ has highest weight λ and is also known as tilting module. Via the Schur functor, $T(\lambda)$ corresponds to the projective $\mathcal{S}_k$-module $P(\lambda)$ with simple quotient D^λ labelled by the partition λ.

Theorem 1 is now an immediate consequence of Proposition 22 and Theorem 11, applied to $U = L_k$.

Remark 23. We have assumed that the dimension r of V should be $\geq pk$. One might ask what can be deduced by truncating the exact sequence

$$0 \to L_{pk}(V) \to e_n \otimes_{F\mathcal{S}_n} V^{\otimes n} \to S^p(L_k(V)) \to 0.$$

See [17, Section 6.5] for details on truncation. Take $d < r$ and take a subspace E of V of dimension d (with basis a subset of the canonical basis of V). There is an idempotent $\zeta \in S(r,n)$ such that $\zeta S(r,n)\zeta \cong S(d,n)$. The functor $(-)\zeta$ is exact. It takes $V^{\otimes n}$ to $E^{\otimes n}$ and $L_{pk}(V)$ to $L_{pk}(E)$. So we get an exact sequence

$$0 \to L_{pk}(E) \to e_n \otimes_{F\mathcal{S}_n} E^{\otimes n} \to S^p(L_k(V))\zeta \to 0.$$

However, $E^{\otimes n}$ is not projective and injective in general as a module for $S(d,n)$. Moreover, the kernel of this sequence can sometimes be isomorphic to an indecomposable summand of $E^{\otimes n}$.

For example, let $p = 3 = n$ and $r = 2$. As mentioned in [13, Example 3.6], the module $L_3(E)$ is isomorphic to $L_2(E) \otimes E$ and is therefore isomorphic to a summand of $E^{\otimes 3}$. (In fact, one can show that this summand is neither projective nor injective as a module for the Schur algebra $S(2, 3)$.)

In concluding this section, we recover a result of Bryant and Stöhr on the pth Lie power.

Let $L(V)''$ denote the second derived algebra of $L(V)$, then the quotient $M(V) = L(V)/L(V)''$ is a free metabelian Lie algebra. It was shown in [11] that

$$L_p(V) \cong B_p(V) \oplus M_p(V),$$

where $B_p(V) = L(V)'' \cap V^{\otimes p}$ and $M_p(V) = (L_p(V) + L(V)'')/L(V)''$.

Corollary 24 ([6, Theorem 3.1]). *$B_p(V)$ is a summand of $V^{\otimes p}$.*

Proof. Assume first that $r \geq p$ and consider the short exact sequence

$$0 \to M_p(V) \to V \otimes S^{p-1}(V) \to S^p(V) \to 0.$$

(see, for instance, [18]) This is non-split since, for example, the Schur functor takes the middle to the natural permutation module of $\mathcal{S}_p$ which is indecomposable. Therefore $M_p(V)$ is not injective.

By Theorem 1, we know that $L_p(V)$ has a unique indecomposable summand which is not projective and injective. It follows that $B_p(V)$ is injective, hence is a summand of $V^{\otimes p}$.

For $r < p$, apply the truncation method as described in the previous remark.

□

7 Factorisation of Lie resolvents

Let G be a group and F be a field. The Green ring R_{FG} of G over F has basis the isomorphism classes of finite-dimensional indecomposable FG-modules with addition and multiplication arising from direct sums and tensor products. If V is a finite-dimensional FG-module, we also write V for the corresponding element in R_{FG}. So, for instance, $V^n \in R_{FG}$ is the isomorphism class of the nth tensor power of V.

If V is a finite-dimensional FG-module, then the nth Lie power $L_n(V)$ may be regarded as a module for FG in a natural way. Recently, Bryant [7, 8, 9] introduced the *Lie resolvents* $\phi^n_{FG} : R_{FG} \to R_{FG}$, $n \geq 1$, to study the structure of $L_n(V)$. These can be described by

$$\phi^n_{FG}(V) = \sum_{d\,|\,n} \mu(n/d) d L_d(V^{n/d})$$

for all n and V, where μ denotes the Möbius function. By Möbius inversion, this is equivalent to

$$L_n(V) = \frac{1}{n} \sum_{d\,|\,n} \phi^d_{FG}(V^{n/d})$$

for all n and all modules V. Thus complete knowledge of the Lie resolvents ϕ^n_{FG} yields a description of $L_n(V)$ for each V, up to isomorphism.

Most strikingly, the Lie resolvents are *linear endomorphisms* of R_{FG} (see [8, Corollary 3.3]). Let p denote the characteristic of F (which may or may not be zero), then $\phi^{k'k}_{FG} = \phi^{k'}_{FG} \circ \phi^k_{FG}$ for all coprime positive integers k, k' not divisible by p (see Theorem 5.4 and Corollary 6.2 in [8]). If the characteristic p of F is positive and G is finite with Sylow p-subgroups of order p, then there is also the identity

$$\phi^{p^m k}_{FG} = \phi^{p^m}_{FG} \circ \phi^k_{FG}$$

for all positive integers m, k such that k is not divisible by p (see [7, Corollary 1.2]). The question was raised in [7] whether such factorisation rule

might hold for arbitrary groups G.

We shall give here an answer in case $m = 1$.

Theorem 25. *Let G be a group and F be a field of prime characteristic p. Then, for any positive integer k not divisible by p,*

$$\phi^{pk}_{FG} = \phi^{p}_{FG} \circ \phi^{k}_{FG}.$$

Proof. Let $n = pk$. It suffices to prove

$$\phi^{pk}_{FG}(V) = \phi^{p}_{FG}(\phi^{k}_{FG}(V)) \tag{15}$$

for any finite-dimensional FG-module V, by linearity. In fact, it suffices to consider the case where F is infinite, $G = \mathrm{GL}(V)$ and V has dimension $r \geq n$, by [7, Lemma 2.4] and arguments completely analogous to those given in the proof of [8, Theorem 5.4]. We write ϕ for ϕ_{FG}.

The equality (15) holds if F has characteristic zero, by Corollary 6.2 and Theorem 5.4 in [8]. In particular, $\phi^{pk}(V)$ and $\phi^{p}(\phi^{k}(V))$ have the same formal character.

We shall now show (15) for all V of dimension $r \geq n$, by induction on k. For $k = 1$, this follows from $\phi^1(V) = L_1(V) = V$. Let $k > 1$, then inductively

$$\begin{aligned}
nL_n(V) &= \sum_{d\,|\,k} (\phi^{pd}(V^{k/d}) + \phi^{d}(V^{pk/d})) \\
&= \phi^{pk}(V) + \phi^{k}(V^p) + \sum_{d\,|\,k,\, d\neq k} (\phi^{pd}(V^{k/d}) + \phi^{d}(V^{pk/d})) \\
&= \phi^{pk}(V) - \phi^{p}(\phi^{k}(V)) + \phi^{p}\Big(\sum_{d\,|\,k} \phi^{d}(V^{k/d})\Big) + \sum_{d\,|\,k} \phi^{d}(V^{pk/d}) \\
&= \phi^{pk}(V) - \phi^{p}(\phi^{k}(V)) + nL_p(L_k(V)) - kL_k(V)^p + kL_k(V^p).
\end{aligned}$$

As a consequence, $\phi^{pk}(V) - \phi^{p}(\phi^{k}(V))$ is a linear combination of $L_n(V) - L_p(L_k(V))$, $L_k(V^p)$ and $L_k(V)^p$ in R_{FG}. The last two are summands of $V^{\otimes n}$ since p does not divide k.

Concerning the first one, note that $L_p(L_k(V))$ embeds naturally into $L_n(V)$. By Theorem 20, there are the two short exact sequences of $\mathrm{GL}(V)$-modules

$$0 \longrightarrow L_p(L_k(V)) \longrightarrow T_1 \longrightarrow S^p(L_k(V)) \longrightarrow 0$$

and

$$0 \longrightarrow L_n(V) \longrightarrow T_2 \longrightarrow S^p(L_k(V)) \longrightarrow 0,$$

where $T_1 = e_p \otimes_{F\mathcal{S}_p} (L_k(V))^{\otimes p}$ and $T_2 = e_n \otimes_{F\mathcal{S}_n} V^{\otimes n}$. The modules T_1 and T_2 are summands of $V^{\otimes n}$, thus they are projective as modules for the Schur algebra $S(r,n)$. Applying Schanuel's Lemma, we have $L_p(L_k(V)) + T_2 = T_1 + L_n(V)$ in R_{FG}.

This shows that $\phi^{pk}(V) - \phi^p(\phi^k(V))$ is an integer linear combination of summands of $V^{\otimes n}$, which allows us to deduce (15) from the equality of the corresponding characters [13]. □

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